The relativistic Galperin billiard: π becomes a complete elliptic integral

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Abstract

In Galperin's billiard, a block of mass M sliding towards a block of unit mass resting against a rigid wall undergoes a number of perfectly elastic impacts whose digits are the digits of π when $M = 100^n$. Every treatment of the problem states that relativistic effects are neglected, and at least one published account notes that ordinary initial speeds would provoke "catastrophic corrections from special relativity"; to our knowledge the corrected count has never been computed. We compute it. Working in the scaling limit $\beta \to 0$, $M \to \infty$ with $w = \beta\sqrt{M}$ fixed, we show that the total number of impacts obeys

$$\frac{N}{\sqrt{M}} \longrightarrow f(w) = \frac{4\,K(k)}{\sqrt{w^2+4}}, \qquad k = \frac{w}{\sqrt{w^2+4}},$$

where K is the complete elliptic integral of the first kind. The classical result is recovered exactly: $f(0) = \pi$. The leading correction is $N \simeq \pi\sqrt{M}\,(1 - \beta^2 M/16)$, which fixes the initial speed at which the d-th digit of π is lost. Theory and an exact event-driven relativistic simulation agree to six significant figures over four decades in M. We note that the derivation never uses relativity as such, only the dispersion relation of the light body, and we record the resulting general statement together with its relation to the adiabatic-invariance literature, where it is likely to be less new.

1 Introduction

Consider a rigid wall, a block of mass $m = 1$ initially at rest against it, and a block of mass $M > m$ sliding towards the pair. All collisions — block–block and block–wall — are perfectly elastic, the motion is frictionless and confined to a line. Galperin [1] proved that the total number of impacts before the blocks separate for good is

$$N = \left\lceil \frac{\pi}{\theta} \right\rceil - 1, \qquad \theta = \arctan\sqrt{m/M}, \tag{1}$$

so that for $M/m = 100^n$ one obtains $3, 31, 314, 3141, \ldots$: an integer count reproduces the decimal digits of π. The result became widely known through an expository video series [2], and it has since been extended in several directions: an explicit solution of the trajectories, a mapping onto a Calogero-type model and a superintegrability statement [3]; a rederivation by adiabatic perturbation theory [4]; a quantum version in which the digits appear in scattering phase shifts [5]; a connection to Grover search [6]; an N-body count in terms of angles between collision subspaces [7]; a family of circular billiards that compute Legendre's elliptic integrals from prescribed initial data [8]; and a set of reflection methods on conics, in which a deliberately non-physical "hyperbolic kinetic energy" $\frac{1}{2}(MV^2 - mv^2)$ produces $\ln 2$ in place of π [9, 10].

All of this is Newtonian. The idealisation is always stated and never lifted. The sharpest statement of the gap we have found is in Brown [6], who observes that ordinary initial velocities provoke "catastrophic corrections from special relativity", and then proceeds without computing them. This is a reasonable place to stop — the correction sounds like it should destroy the structure — but the expectation turns out to be wrong. Relativity does not destroy the count. It replaces π by a complete elliptic integral of the first kind, of which π is the exact zero-velocity limit.

Section 2 fixes the relativistic dynamics and reports a numerical scaling law. Section 3 derives the closed form. Section 4 states the result, its expansions, and the numerical check. Section 5 records the general dispersion-relation statement that the derivation actually proves, and is explicit about which parts of it are likely to be known already.

2 Relativistic dynamics and a scaling law

We set $c = 1$. The wall is rigid and infinitely massive, so a wall impact reverses the light block's velocity exactly. For a block–block impact we use the exact one-dimensional relativistic elastic collision, most simply expressed by boosting to the centre-of-momentum frame, reversing both velocities there, and boosting back. With $\gamma(u) = (1 - u^2)^{-1/2}$ and pre-collision velocities u_1, u_2,

$$V = \frac{\gamma(u_1)u_1 + \gamma(u_2)Mu_2}{\gamma(u_1) + \gamma(u_2)M}, \qquad u_i' = \frac{-\dfrac{u_i - V}{1 - u_i V} + V}{1 - \dfrac{u_i - V}{1 - u_i V}V}. \tag{2}$$

The initial condition is $u_1 = 0$, $u_2 = -\beta$. The process terminates when the blocks can no longer meet.

Iterating (2) in high-precision decimal arithmetic reproduces the classical counts $3, 31, 314, 3141$ as $\beta \to 0$, as it must. Raising β destroys them; the interesting observation is *how*. Table 1 shows the count at three mass ratios and three initial speeds chosen so that $\beta\sqrt{M}$ is constant along each row.

$\beta\sqrt{M}$	$M = 10^4$	$M = 10^6$	$M = 10^8$
0.1	313	3 139	31 396
1	296	2 968	29 688
10	118	1 190	11 904

Table 1: Total impacts. Along each row the counts are the same digits shifted by one decimal place, so $N/\sqrt{M}$ depends on β and M only through the combination $w = \beta\sqrt{M}$. (Classical values: 314, 3141, 31415.)

This is the whole structure of the problem in one table. We therefore work in the limit

$$\beta \to 0, \qquad M \to \infty, \qquad w = \beta\sqrt{M} \ \text{ fixed}, \tag{3}$$

in which the heavy block remains non-relativistic while carrying a fixed kinetic energy $w^2/2$, and the light block becomes fully relativistic: relativity enters exactly when the light block would classically be required to exceed c, i.e. when $w \gtrsim 1$.

3 Derivation

Let p be the momentum of the light block and P that of the heavy one, and set $X = P/\sqrt{M}$. In the limit (3) the heavy block contributes kinetic energy $X^2/2$, so the conserved energy curve in the (X, p) plane is

$$\frac{X^2}{2} + \sqrt{1+p^2} \;=\; 1 + \frac{w^2}{2}, \tag{4}$$

a smooth convex oval, symmetric in p, with $|X| \le w$. Conservation of total momentum reads $\sqrt{M}\,X + p = \text{const}$, so a block–block collision moves the state along a chord of fixed direction $(1, -\sqrt{M})$ — nearly parallel to the p-axis — to the second intersection of that chord with (4). A wall impact is the exact reflection $p \mapsto -p$, which preserves the oval.

Because the chord is nearly vertical, a block–block collision sends $p \mapsto -p$ up to a correction of order $M^{-1/2}$, and the wall impact restores the sign. One pair of impacts therefore returns p to its value and advances X by

$$\Delta X = \frac{2p}{\sqrt{M}} + O(M^{-1}). \tag{5}$$

The number of pairs required to sweep X across the oval is $\int \mathrm{d}X/\Delta X$, and each pair contributes two impacts, so

$$\boxed{N = \sqrt{M} \int_{-w}^{w} \frac{\mathrm{d}X}{p(X)}} \tag{6}$$

with $p(X) \ge 0$ determined by (4). Solving (4) for p,

$$p(X) = \tfrac{1}{2}\sqrt{\left(w^2 - X^2\right)\left(4 + w^2 - X^2\right)}, \tag{7}$$

and substituting $X = w \sin t$ turns (6) into

$$\frac{N}{\sqrt{M}} = 4\int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{w^2 + 4 - w^2 \sin^2 t}} = \frac{4}{\sqrt{w^2+4}}\, K\!\left(\frac{w}{\sqrt{w^2+4}}\right), \tag{8}$$

where $K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 t)^{-1/2} \mathrm{d}t$.

Proposition 1. *In the scaling limit* (3)*, the total number of impacts in the relativistic Galperin billiard satisfies*

$$\frac{N}{\sqrt{M}} \;\longrightarrow\; f(w) = \frac{4\,K(k)}{\sqrt{w^2+4}}, \qquad k = \frac{w}{\sqrt{w^2+4}}. \tag{9}$$

Consistency check. For a non-relativistic light block one replaces $\sqrt{1+p^2}$ by $1 + p^2/2$ in (4); the oval becomes the circle $X^2 + p^2 = w^2$ and (6) gives $\int_{-w}^{w} \mathrm{d}X/\sqrt{w^2 - X^2} = \pi$, i.e. Galperin's result. The circle hiding in the classical problem is thus the $w \to 0$ shape of the oval (4), and the appearance of π is the statement that a circle has angular measure π in a half-turn.

4 Results

Classical limit. $k \to 0$ and $K(0) = \pi/2$, so

$$f(0) = \frac{4 \cdot (\pi/2)}{2} = \pi \tag{10}$$

exactly. Galperin's constant is the zero-velocity value of (9).

Leading correction. Expanding K and the prefactor,

$$f(w) = \pi \left(1 - \frac{w^2}{16} + O(w^4)\right), \qquad \text{i.e.} \qquad N \simeq \pi\sqrt{M}\left(1 - \frac{\beta^2 M}{16}\right). \tag{11}$$

Since the digits of π are read off from N, the d-th digit survives only while the relative deficit stays below 10^{-d}, that is while

$$\beta^2 M \lesssim 16 \times 10^{-d}. \tag{12}$$

Reading d digits therefore requires an initial speed below $\beta \sim 4 \cdot 10^{-d/2}/\sqrt{M}$: the classical experiment is not merely idealised, it is quantitatively constrained.

Ultra-relativistic regime. As $w \to \infty$, $k \to 1$ and $K(k) \simeq \ln\bigl(2\sqrt{w^2+4}\bigr)$, so

$$f(w) \simeq \frac{4\ln(2w)}{w}, \tag{13}$$

a logarithmically slow decay rather than an abrupt collapse.

Numerical verification. Table 2 compares (9) against the exact event-driven simulation (2) run in 50–60 digit decimal arithmetic. K is evaluated by the arithmetic–geometric mean, $K(k) = \pi/\bigl(2\,\mathrm{agm}(1, \sqrt{1-k^2})\bigr)$.

w	$f(w)$ (theory)	$M = 10^6$	$M = 10^8$	$M = 10^{10}$	$M = 10^{12}$
0.1	3.139632	3.13900	3.13960	3.13963	3.139631
0.3	3.124141	3.12400	3.12410	3.12414	3.124141
1.0	2.968825	2.96800	2.96880	2.96882	2.968824
2.0	2.622058	2.62200	2.62200	2.62205	2.622057
5.0	1.803351	1.80300	1.80330	1.80335	1.803350
10.0	1.190470	1.19000	1.19040	1.19047	1.190470
30.0	0.545455	0.54500	0.54540	0.54545	0.545455

Table 2: $N/\sqrt{M}$ from the exact relativistic simulation, against (9). The residual at finite M is $O(M^{-1/2})$, consistent with the neglected term in (6).

5 What the derivation actually proves

Relativity is used nowhere in Section 3 except through the particular energy–momentum relation of the light body. Replacing $\sqrt{1+p^2} - 1$ by an arbitrary dispersion $\varepsilon(p)$, even and increasing in $|p|$, the same argument gives

$$\frac{N}{\sqrt{M}} \longrightarrow \int \frac{\mathrm{d}X}{p(X)}, \qquad \varepsilon\bigl(p(X)\bigr) = \frac{X_0^2 - X^2}{2}, \tag{14}$$

X_0 being the initial value of $|X|$. Three consequences are worth recording.

(i) π is a low-energy universal. Every smooth dispersion is parabolic near its minimum, so (14) tends to π as $X_0 \to 0$ whatever ε is. What the count measures at higher energy is the departure from parabolicity: writing $\varepsilon(p) = p^2/2 + a_4 p^4 + \dots$ one finds $f \simeq \pi(1 + a_4 X_0^2/2)$. For the relativistic dispersion $a_4 = -1/8$, recovering (11); for a tight-binding band $\varepsilon = 1 - \cos p$, $a_4 = -1/24$.

(ii) A third worked case. Formula (14) has been checked against simulation for $\varepsilon = p^2/2$, $\varepsilon = \sqrt{1+p^2} - 1$ and $\varepsilon = 1 - \cos p$ at several energies and two mass ratios; the residual at

$M = 10^8$ is consistent with the expected $O(M^{-1/2})$ error in every case (see the accompanying code).

(iii) An inverse problem. Changing variables, (14) reads $f(E) = \sqrt{2}\int_0^E p(\epsilon)^{-1}(E-\epsilon)^{-1/2}\mathrm{d}\epsilon$, an Abel transform, so measuring the count at several energies determines $\varepsilon(p)$.

We flag (i)–(iii) as considerably less likely to be new than (9). The adiabatic invariant of this exact system is the subject of [4, 12], and (14) is essentially an action integral; point (iii) is the classical Abel inversion of a period integral, standard since [13] and treated as a textbook exercise. The realisability question attached to it — which count profiles arise from convex dispersions — belongs to the well-developed theory of period functions [14, 15]. We record these statements for completeness and make no priority claim on them.

6 Relation to previous work

Our result is disjoint from the existing extensions of Galperin's billiard, and it is worth being precise about why.

Dubeau [9, 10] obtains $\ln 2$ and inverse hyperbolic functions by replacing the conserved quantities with $Q = MV - mv$ and $E = \frac{1}{2}(MV^2 - mv^2)$, differences rather than sums, which he describes as a hypothetical hyperbolic setting; his framework is confined to the conics $x^2 \pm y^2 = 1$ and yields elementary functions only. Special relativity conserves $\sqrt{m_1^2 + p_1^2} + \sqrt{m_2^2 + p_2^2}$, a sum, whose level set is a quartic oval and not a conic; this is precisely why an elliptic integral appears where Dubeau finds a logarithm.

Melnik [8] computes Legendre's elliptic integrals from a family of *circular* billiards with non-interacting balls, encoding the desired integral in the initial data and extracting it through Legendre and Gauss identities; he treats Galperin's method as a separate example. In (9) the elliptic integral is not encoded but imposed: nothing is chosen, and the modulus k is fixed by the kinematics.

Cai and Zhang [5] replace the light body by a quantum particle; Ee, Kim and Lee [11] solve a relativistic one-dimensional system in closed form, but with a massless mediator shuttling between two masses and no wall, and their question is the effective interaction law rather than a collision count. Aretxabaleta et al. [3] generalise the numerical *base* of the count, including irrational bases, and analyse the systematic errors of the classical scheme; those errors are the truncation of $\arctan x \approx x$ and are unrelated to (11).

A literature search covering the citation neighbourhood of [1] and [3] did not locate a relativistic treatment of the count. We note that this search could not be made exhaustive.

7 Conclusion

The count of impacts in Galperin's billiard is not attached to π; it is attached to the shape of the conserved energy curve, and π is what that shape gives when the light body is Newtonian. Special relativity bends the curve from a circle into a quartic oval and replaces π by $4K(k)/\sqrt{w^2+4}$, continuously and with no loss of exactness. The classical constant is recovered as $w \to 0$, and the rate at which the digits of π are destroyed is $\beta^2 M/16$.

Whether the count can be realised at a mass ratio and precision permitting even a handful of impacts in an actual experiment [16] is a separate question, and we do not claim that it can.

Code and data. The event-driven relativistic simulator, the arithmetic–geometric-mean evaluation of K, the generalised-dispersion check and the Abel inversion are provided with this note, in Python 3 with no external dependencies.

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